

Orthogonal Projections Based on Hyperbolic and Spherical n -Simplex

Baki Karlığa¹, Murat Savaş¹ and Atakan T. Yakut²

¹ Gazi University, Faculty of Science, Department of Mathematics

06500, Ankara-Turkey

² Nigde University, Art and Sciences Faculty, Department of Mathematics

51100, Nigde-Turkey

e-mail: karliaga@gazi.edu.tr, msavas@gazi.edu.tr, sevaty@nigde.edu.tr

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Abstract

In this paper, orthogonal projection along a geodesic to the chosen k -plane is introduced using edge and Gram matrix of an n -simplex in hyperbolic or spherical n -space. The distance from a point to k -plane is obtained by the orthogonal projection. It is also given the perpendicular foots from a point to k -plane of hyperbolic and spherical n -space.

1 Introduction

One of the fundamental notions in geometry is orthogonal projection and also studies extensively through the long history of mathematics and physics. There are many applications of orthogonal projection. The concept of orthogonal projection plays an important role in the scattering theory, the theory of many-body resonance and different branches of theoretical and mathematical physics.

Let R_1^{n+1} be $(n + 1)$ -dimensional vector space equipped with the scalar

product \langle, \rangle which is defined by

$$\langle x, y \rangle = -x_1 y_1 + \sum_{i=2}^{n+1} x_i y_i.$$

If the restriction of scalar product on a subspace W of R_1^{n+1} is positive definite, then the subspace W is called *space-like*; if it is positive semi-definite and degenerate, then W is called *light-like*; if W contains a time-like vector of R_1^{n+1} , then W is called *time-like*.

$S_1^n = \{x \in R_1^{n+1} \mid \langle x, x \rangle = 1\}$ is called *de Sitter n -space*. The n -dimensional *unit pseudo-hyperbolic space* is defined as

$$H_0^n = \{x \in R_1^{n+1} \mid \langle x, x \rangle = -1\},$$

which has two connected components, each of which can be considered as a model for the n -dimensional *hyperbolic space* H^n . Throughout this paper we consider *hyperbolic n -space*

$$H^n = \{x \in H_0^n \mid x_1 > 0\}.$$

Hence, each pair of points p_i, p_j in H^n satisfy $\langle p_i, p_j \rangle < 0$. The *hyperbolic distance* for $p, q \in H^n$ is defined by $\text{arccosh}(-\langle p, q \rangle)$. Since each $e \in S_1^n$ determines a time-like hyperplane of R_1^{n+1} , we have hyperplane $e^\perp \cap H^n$ of H^n .

Let R^{n+1} be $(n+1)$ -dimensional vector space equipped with the scalar product \langle, \rangle_E which is defined by

$$\langle x, y \rangle_E = \sum_{i=1}^{n+1} x_i y_i.$$

The n -dimensional *unit spherical space* S^n is given by

$$S^n = \{x \in R^{n+1} \mid \langle x, x \rangle_E = 1\}.$$

The spherical distance $d_s(p, q)$ between p and q is given by $\arccos(\langle p, q \rangle_E)$.

We consider W is a vector subspace spanned by the vectors e_1, e_2, \dots, e_{n-k} in S_1^n . By using Lemma 27 of [1], one can easily see that W is $(n-k)$ -dimensional time-like subspace and $V = e_1^\perp \cap e_2^\perp \cap \dots \cap e_{n-k}^\perp$ is $(k+1)$ -dimensional time-like subspace of R_1^{n+1} . Consequently, for $i = 1, 2, \dots, n-k$, the hyperplane $e_i^\perp \cap H^n$ intersect at the time-like k -plane $V \cap H^n$ of H^n . One can define the same tools for spherical n -space.

Let \triangle be a hyperbolic or spherical n -simplex with vertices p_1, \dots, p_{n+1} , and \triangle_i be the face opposite to vertex p_i . Then, according to the first section of [2], we have the edge matrix M and Gram matrix G of \triangle . Let $|M|$ and M_{ij} be the determinant and ij th-minor of M , then the unit outer normal vector of \triangle_i is given by

$$e_i = \frac{-\epsilon \sum_{j=1}^{n+1} M_{ij} p_j}{\sqrt{M_{ii}|M|}}, \quad i = 1, \dots, n+1,$$

where ϵ is the curvature of space.

The intersection of H^n with $(k+1)$ -dimensional time-like subspace is called k -dimensional plane of H^n [3]. Similarly, a k -plane of spherical space is given by the same way.

When a geodesic is drawn orthogonally from a point to a k -plane, its intersection with the k -plane is known as *perpendicular foot* on that k -plane in H^n or S^n . The length of a geodesic segment bounded by a point and its perpendicular foot is called *the distance between that point and k -plane*. The distance between a vertex and its any opposite k -face is called *k -face altitude* of an n -simplex.

The orthogonal projection to 2-plane in Euclidean space is well-known (see [4],[5],[6],[7]). The orthogonal projection to k -plane in Euclidean space is given in [8]. The orthogonal projection taking a point in H^n and mapping it to its perpendicular foot on a hyperplane are studied in [3] and [9], respectively. The distance between a point and a hyperbolic(spherical) hyperplane is introduced in [10]. The altitude of $(n-1)$ -face of hyperbolic n -simplex is given in [11].

The orthogonal projection taking a point along a geodesic and mapping to its perpendicular foot, where geodesic meets orthogonally the chosen k -plane of projection, has not been studied. The aim of this paper is to study such orthogonal projections according to the edge matrix of a simplex in H^n or S^n .

Let m^{k+1} be the determinant of sub-matrix $M(k+1, \dots, n+1)$ of M and g^{k+1} be the determinant of sub-matrix $G(k+1, \dots, n+1)$ of G . Suppose that m_i^j and g_i^j be the determinant of sub-matrix $M \begin{pmatrix} 1 & \dots & k+1 & i \\ 1 & \dots & k+1 & j \end{pmatrix}_{i,j=k+2, \dots, n+1}$ and $G \begin{pmatrix} 1 & \dots & k+1 & i \\ 1 & \dots & k+1 & j \end{pmatrix}_{i,j=k+2, \dots, n+1}$, respectively.

Lemma 1.1 *Let \triangle be a hyperbolic or spherical n -simplex with the edge matrix M and Gram matrix G . Let M_{ii} and G_{ii} be i th minor of M and G , respectively. Then $M^{-1} = TGT$ and $G^{-1} = TMT$*

$$\text{where } T = \left[\sqrt{\frac{G_{ii}}{\epsilon|G|}} \delta_{ij} \right] = \left[\sqrt{\frac{M_{ii}}{\epsilon|M|}} \delta_{ij} \right].$$

Proof It can be seen from [12]. \square

Let M^{11}, M^{12}, M^{22} and G^{11}, G^{12}, G^{22} be $(k+1) \times (k+1)$, $(k+1) \times (n-k)$, $(n-k) \times (n-k)$ types sub-matrices of M and G , respectively. Suppose that M, G and diagonal matrix T partitioned as $\begin{bmatrix} M^{11} & M^{12} \\ M^{12} & M^{22} \end{bmatrix}$, $\begin{bmatrix} G^{11} & G^{12} \\ G^{12} & G^{22} \end{bmatrix}$ and $\begin{bmatrix} T^{11} & 0 \\ 0 & T^{22} \end{bmatrix}$, respectively.

Concerning Lemma 1.1 along with Schur complement of a symmetric matrix, we have the following lemma.

Lemma 1.2 *Let $S_{M^{ii}}$ and $S_{G^{ii}}$ be Schur complements of the sub-matrices M^{ii} and G^{ii} . Then*

$$(M^{ii})^{-1} = T^{ii} S_{G^{jj}} T^{ii}, \quad (G^{ii})^{-1} = T^{ii} S_{M^{jj}} T^{ii} \quad i \neq j; \quad i, j = 1, 2.$$

Proof It is obvious that M, G are symmetric and M^{ii}, G^{ii} are invertible. Since the inverse of Schur complement of M^{11} in M is the sub-matrix of M^{-1} , we have

$$M^{-1} = \begin{bmatrix} (M^{11})^{-1} + (M^{11})^{-1} M^{12} (S_{M^{11}})^{-1} M^{21} (M^{11})^{-1} & - (M^{11})^{-1} M^{12} (S_{M^{11}})^{-1} \\ - (S_{M^{11}})^{-1} M^{21} (M^{11})^{-1} & (S_{M^{11}})^{-1} \end{bmatrix}.$$

Similarly for the Schur complement of M^{22} , we obtain

$$M^{-1} = \begin{bmatrix} (S_{M^{22}})^{-1} & - (S_{M^{22}})^{-1} M^{12} (M^{22})^{-1} \\ - (M^{22})^{-1} M^{21} (S_{M^{22}})^{-1} & (M^{22})^{-1} + (M^{22})^{-1} M^{21} (S_{M^{22}})^{-1} M^{12} (M^{22})^{-1} \end{bmatrix}$$

Then we have

$$M^{-1} = \begin{bmatrix} (S_{M^{22}})^{-1} & - (M^{11})^{-1} M^{12} (S_{M^{11}})^{-1} \\ - (M^{22})^{-1} M^{21} (S_{M^{22}})^{-1} & (S_{M^{11}})^{-1} \end{bmatrix},$$

by the same way, we get

$$G^{-1} = \begin{bmatrix} (S_{G^{22}})^{-1} & -(G^{11})^{-1} G^{12} (S_{G^{11}})^{-1} \\ -(G^{22})^{-1} G^{21} (S_{G^{22}})^{-1} & (S_{G^{11}})^{-1} \end{bmatrix}.$$

Thus, we obtain the desired results using Lemma 1.1. \square

2 Orthogonal Projection to k -plane of H^n based on a Hyperbolic n -simplex

If p_1, p_2, \dots, p_{n+1} are vertices of any hyperbolic n -simplex \triangle , then $\{p_1, p_2, \dots, p_{n+1}\}$ is a basis of R_1^{n+1} . Let W_j be a subspace spanned by $\{p_1, p_2, \dots, \hat{p}_j, \dots, p_{n+1}\}$, and e_j be the unit outer normal to W_j ; $j = 1, \dots, n+1$. Hence $\{e_1, e_2, \dots, e_{n+1}\}$ is another basis of R_1^{n+1} .

Let W^h be a k -plane which contains a k -face with vertices p_1, p_2, \dots, p_{k+1} of \triangle . Then the set $\{p_1, p_2, \dots, p_{k+1}\}$ is a basis of the $(k+1)$ -dimensional subspace of W in R_1^{n+1} . Since p_1, p_2, \dots, p_{k+1} are vertices of \triangle , the subset $\{p_1, p_2, \dots, p_{k+1}\}$ of R_1^{n+1} can be extended basis $\{p_1, p_2, \dots, p_{n+1}\}$ and $\{p_1, \dots, p_{k+1}, e_{k+2}, \dots, e_{n+1}\}$ of R_1^{n+1} . As a consequence, we see that $\{e_{k+2}, \dots, e_{n+1}\}$ is a basis of $(n-k)$ -dimensional subspace W^\perp .

Theorem 2.1 *Let p be a point and W^h be a k -plane in H^n . Then the orthogonal projection $\sigma(p)$ of p to W^h is given by*

$$\sigma(p) = \frac{p + \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle e_s}{|M|m^{k+1}}}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle \langle p, e_s \rangle}{|M|m^{k+1}}}}.$$

Proof For a point $p \in H^n$, by [10, Theorem 3.11], there is a point $\dot{p} \in W$ such that $\vec{p\dot{p}} \in W^\perp$. Therefore, we can write

$$\vec{p\dot{p}} = \sum_{s=k+2}^{n+1} \lambda_s e_s. \quad (1)$$

Then, we have $-\langle p, e_t \rangle = \sum_{s=k+2}^{n+1} \lambda_s \langle e_s, e_t \rangle$, $t = k+2, \dots, n+1$.

Taking

$$G^{22} = G \begin{pmatrix} k+2 & \dots & n+1 \\ k+2 & \dots & n+1 \end{pmatrix} = [\langle e_s, e_t \rangle]_{s,t=k+2,\dots,n+1}, \quad L = [\lambda_{k+2} \dots \lambda_{n+1}],$$

we obtain

$$L = -(G^{22})^{-1} [\langle p, e_t \rangle]. \quad (2)$$

By Lemma 1.2 and the equation (5) of [14], we see that

$$(G^{22})^{-1} = \left[\frac{\sqrt{M_{ii}M_{jj}m_j^i}}{-|M|m^{k+1}} \right]_{i,j=k+2,\dots,n+1},$$

and this implies

$$\lambda_s = \sum_{t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_t^s} \langle p, e_t \rangle}{|M|m^{k+1}}, \quad s = k+2, \dots, n+1. \quad (3)$$

Substituting (3) into (1), we obtain

$$\dot{p} = p + \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_t^s} \langle p, e_t \rangle e_s}{|M|m^{k+1}}. \quad (4)$$

By [10], there exists a unique $\sigma(p) \in W^h$ such that $\sigma(p) = c\dot{p}$. Since \dot{p} is the orthogonal projection of p to W , we have

$$c = \frac{1}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}m_t^s} \langle p, e_t \rangle \langle p, e_s \rangle}{|M|m^{k+1}}}}}$$

which completes the proof. \square

In case of the orthogonal projection to a hyperplane, we obtain $m_j^j = |M|$ and $m^{k+1} = M_{jj}$. Substituting these equalities into the statement of Theorem 2.1, we reach the result of [3, Theorem 4.1] and [13, Proposition 2.2], as follows:

$$\sigma(p) = \frac{p - \langle p, e_j \rangle e_j}{\sqrt{1 + \langle p, e_j \rangle^2}}, \quad j = 1, \dots, n+1$$

where e_j is the unit normal of W_j in R_1^{n+1} . This result is also a generalization of Theorem 2.1 [3, 13].

Theorem 2.2 *Let p be a point and W^h be a k -plane in H^n . Then,*

$$\cosh \xi(p, W^h) = \sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle \langle p, e_s \rangle}{|M|m^{k+1}}}.$$

where $\xi(p, W^h)$ is the distance between p and W^h .

Proof Since $\langle p, \sigma(p) \rangle = -\cosh \xi(p, W^h)$, the result follows Theorem 2.1. \square

As an immediate consequence of Theorem 2.2, we obtain the following known result[10, Section 4].

Corollary 2.3 *Let p be a point and W_j^h be a hyperplane of H^n determined by e_j . Then the distance $\xi(p, W_j^h)$ between p and W_j^h is given by*

$$\cosh \xi(p, W_j^h) = \sqrt{1 + \langle p, e_j \rangle^2}.$$

By taking p_j instead of p in (4) and using $\langle p_j, e_t \rangle = -\sqrt{\frac{|M|}{M_{tt}}} \delta_{jt}$, we obtain

$$\dot{p}_j = p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m_j^s}{m^{k+1}} e_s$$

and

$$\langle \dot{p}_j, \dot{p}_j \rangle = -1 - \frac{m_j^j}{m^{k+1}}$$

where p_j is a vertex of \triangle . The proof of following corollary is obvious from Theorem 2.1.

Corollary 2.4 *Let \triangle be a hyperbolic simplex with vertices p_1, \dots, p_{n+1} . Then the perpendicular foot from p_j to k -face W^h is given by*

$$\sigma(p_j) = \frac{p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m_j^s}{m^{k+1}} e_s}{\sqrt{1 + \frac{m_j^j}{m^{k+1}}}}, \quad j = k+2, \dots, n+1,$$

where p_1, \dots, p_{k+1} are vertices of k -face W^h .

If we replace p by p_j and use $\langle p_j, e_t \rangle = -\sqrt{\frac{|M|}{M_{tt}}} \delta_{jt}$, we see that

$$\langle \sigma(p_j), p_j \rangle = -\sqrt{1 + \frac{m_j^j}{m^{k+1}}}.$$

If we consider the last equation in the proof of Theorem 2.1, we see that $\cosh \xi(p_j, W^h) = \sqrt{1 + \frac{m_j^j}{m^{k+1}}}$, that result is a generalization of [11, Proposition 4] to the k -face W^h of a hyperbolic n -simplex. Since $\frac{m_j^j}{m^{k+1}}$ is the diagonal jj th-entry of $S_{M^{11}} = [a_{ij}]$, the altitude from p_j to k -face W^h with vertices p_1, \dots, p_{k+1} is given by

$$\cosh \xi(p_j, W^h) = \sqrt{1 + a_{jj}}$$

where $\xi(p_j, W^h)$ is the distance between p_j and k -face W^h .

By $\dot{p}_j = p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j$, for $(n-1)$ -face W_j^h , we have the following corollary.

Corollary 2.5 *Let \triangle be a hyperbolic simplex with vertices p_1, \dots, p_{n+1} . Then the perpendicular foot from p_j to $(n-1)$ -face W_j^h is given by*

$$\sigma(p_j) = \frac{p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j}{\sqrt{1 + \frac{|M|}{M_{jj}}}}, \quad j = 1, \dots, n+1$$

where $p_1, \dots, \hat{p}_j, \dots, p_{n+1}$ are vertices of W_j^h .

Using $G_{jj} = \frac{-|G|M_{jj}}{|M|}$ for $j = 1, \dots, n+1$, we obtain the following known result [11, Proposition 4].

Corollary 2.6 *Let \triangle be a hyperbolic simplex with vertices p_1, \dots, p_{n+1} . Then the altitude $\xi(p_j, W_j^h)$ from p_j to $(n-1)$ -face W_j^h is given by*

$$\cosh \xi(p_j, W_j^h) = \sqrt{1 + \frac{|M|}{M_{jj}}}, \quad j = 1, \dots, n+1$$

where $p_1, \dots, \hat{p}_j, \dots, p_{n+1}$ are vertices of W_j^h .

3 Orthogonal Projections to a k -plane of S^n based on a Spherical n -simplex

Let \triangle be with vertices p_1, \dots, p_{n+1} . Then $\{p_1, \dots, p_{n+1}\}$ is a basis of R^{n+1} . If W_j is the subspace spanned by $\{p_1, \dots, \hat{p}_j, \dots, p_{n+1}\}$, then $\{e_1, \dots, e_{n+1}\}$ is another basis of R^{n+1} where e_j is the unit outer normal to W_j for $j = 1, \dots, n+1$.

Let W^s be a k -plane which contains a k -face with vertices p_1, p_2, \dots, p_{k+1} . Then the set $\{p_1, p_2, \dots, p_{k+1}\}$ is a basis of the $(k+1)$ -dimensional subspace W in R^{n+1} . As a consequence, we have a basis $\{e_{k+2}, \dots, e_{n+1}\}$ of $(n-k)$ -dimensional subspace W^\perp .

Theorem 3.1 *Let p be a point and W^s be a k -plane in S^n . Then the orthogonal projection $\sigma(p)$ of p to W^s is given by*

$$\sigma(p) = \frac{p - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E e_s}{|M|m^{k+1}}}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E \langle p, e_s \rangle_E}{|M|m^{k+1}}}}.$$

Proof By [10, Theorem 3.11], for $p \in S^n$, there is a $\vec{p} \in W$ such that $p\vec{p} \in W^\perp$. Therefore, we can write

$$\vec{p\vec{p}} = \sum_{s=k+2}^{n+1} \lambda_s e_s \quad (5)$$

Then, we have

$$\langle p, \vec{p\vec{p}} \rangle_E = \sum_{s=k+2}^{n+1} \lambda_s \langle e_s, e_t \rangle_E.$$

Taking

$$G^{22} = G \begin{pmatrix} k+2 & \dots & n+1 \\ k+2 & \dots & n+1 \end{pmatrix} = [\langle e_s, e_t \rangle_E]_{s,t=k+2, \dots, n+1}, \quad L = [\lambda_{k+2} \dots \lambda_{n+1}],$$

we find

$$L = -(G^{22})^{-1} [\langle p, e_t \rangle_E]. \quad (6)$$

By Lemma 1.2 and the equation (5) of [14], we see that

$$(G^{22})^{-1} = \left[\frac{\sqrt{M_{ii}M_{jj}}m_j^i}{|M|m^{k+1}} \right]_{i,j=k+2, \dots, n+1}.$$

This implies

$$\lambda_s = - \sum_{t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E}{|M|m^{k+1}}, \quad s = k+2, \dots, n+1. \quad (7)$$

Substituting (7) into (5), we obtain

$$\dot{p} = p - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E e_s}{|M|m^{k+1}}. \quad (8)$$

By [10], there exists a unique $\sigma(p) \in W^s$ such that $\sigma(p) = c\dot{p}$. Since \dot{p} is the orthogonal projection of p to W , we have

$$c = \frac{1}{\sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E \langle p, e_s \rangle_E}{|M|m^{k+1}}}}$$

which completes the proof. \square

By Theorem 3.1, we have

$$\sigma(p) = \frac{p - \langle p, e_j \rangle_E e_j}{\sqrt{1 - \langle p, e_j \rangle_E^2}}$$

where e_j is the unit normal of the W_j in R^{n+1} .

Theorem 3.2 *Let p be a point and W^s be a k -plane in S^n . Then*

$$\cos \theta(p, W^s) = \sqrt{1 - \sum_{s,t=k+2}^{n+1} \frac{\sqrt{M_{ss}M_{tt}}m_t^s \langle p, e_t \rangle_E \langle p, e_s \rangle_E}{|M|m^{k+1}}}.$$

where $\theta(p, W^s)$ is the distance between p and W^s .

By taking p_j instead of p and using $\langle p_j, e_j \rangle_E = -\sqrt{\frac{|M|}{M_{jj}}}$ in (8), we obtain

$$\dot{p}_j = p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m_j^s}{m^{k+1}} e_s$$

and

$$\langle \dot{p}_j, \dot{p}_j \rangle_E = 1 - \frac{m_j^j}{m^{k+1}}, \quad j = k+2, \dots, n+1,$$

where p_j is a vertex of \triangle . Hence, we have the following corollary.

Corollary 3.3 *Let \triangle be a spherical n -simplex with vertices p_1, \dots, p_{n+1} , then the perpendicular foot from p_j to k -face W^s is given by*

$$\sigma(p_j) = \frac{p_j + \sum_{s=k+2}^{n+1} \sqrt{\frac{M_{ss}}{|M|}} \frac{m_j^s}{m^{k+1}} e_s}{\sqrt{1 - \frac{m_j^j}{m^{k+1}}}}, \quad j = k+2, \dots, n+1,$$

where p_1, \dots, p_{k+1} are vertices of W^s .

Let $\theta(p_j, W^s)$ be the altitude from the vertex p_j to the k -face W^s with vertices p_1, \dots, p_{k+1} for $j = k+2, \dots, n+1$. Then $\theta(p_j, W^s)$ is given by

$$\cos \theta(p_j, W^s) = \sqrt{1 - \frac{m_j^j}{m^{k+1}}}.$$

By equality (5) in [14], the jj th-entry of the Schur complement $S_{M^{11}} = [b_{ij}]$ satisfy $b_{jj} = \frac{m_j^j}{m^{k+1}}$.

Let W_j^s be the $(n-1)$ -face with vertices $p_1, \dots, \hat{p}_j, \dots, p_{n+1}$ of \triangle . Then, we have

$$\dot{p}_j = p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j,$$

and

$$\langle p_j, e_j \rangle_E = -\sqrt{\frac{|M|}{M_{jj}}}, \quad j = 1, \dots, n+1.$$

The proof of the following corollary is obtained by using the above equations.

Corollary 3.4 *Let \triangle be a spherical simplex with vertices p_1, \dots, p_{n+1} . Then the perpendicular foot from p_j to $(n-1)$ -face W_j^s is given by*

$$\sigma(p_j) = \frac{p_j + \sqrt{\frac{|M|}{M_{jj}}} e_j}{\sqrt{1 - \frac{|M|}{M_{jj}}}}, \quad j = 1, \dots, n+1,$$

where $p_1, \dots, \hat{p}_j, \dots, p_{n+1}$ are vertices of W_j^s .

Corollary 3.5 *Let \triangle be a spherical simplex with vertices p_1, \dots, p_{n+1} . Then the altitude $\theta(p_j, W_j^s)$ from p_j to $(n-1)$ -face W_j^s is given by*

$$\cos \theta(p_j, W_j^s) = \sqrt{1 - \frac{|M|}{M_{jj}}}, \quad j = 1, \dots, n+1.$$

where $p_1, \dots, \hat{p}_j, \dots, p_{n+1}$ are vertices of W_j^s .

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